

On a General Sextic Equation Solved by the Rogers-Ramanujan Continued fraction

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Keywords: Sextic equation; j -invariant; Ramanujan; Continued fraction; Algebraic Equations; Algebraic Numbers; Elliptic Functions; Modular equations;

Abstract

In this article we solve a general class of sextic equations. The solution follows if we consider the j -invariant and relate it with the polynomial equation's coefficients. The form of the solution is a relation of Rogers-Ramanujan continued fraction. The inverse technique can also be used for the evaluation of the Rogers-Ramanujan continued fraction, in which the equation is not now the depressed equation but another quite more simplified equation.

1 Introductory Definitions

We will solve the following equation

$$\frac{b^2}{20a} + bX^3 + aX^6 = C_1X^5 \quad : (eq)$$

or equivalent

$$\frac{b^2}{20a} + bX + aX^2 = C_1X^{5/3} \quad (1)$$

using the j -invariant and the Rogers-Ramanujan continued fraction.

For $|q| < 1$, the Rogers Ramanujan continued fraction (RRCF) (see [2],[3],[4]) is defined as

$$R(q) := \frac{q^{1/5}}{1+} \frac{q^1}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \dots \quad (2)$$

From the Theory of Elliptic functions the j -invariant (see [5],[8]) is

$$j_r := \left[\left(\frac{\eta(\frac{1}{2}\sqrt{-r})}{\eta(\sqrt{-r})} \right)^{16} + 16 \left(\frac{\eta(\sqrt{-r})}{\eta(\frac{1}{2}\sqrt{-r})} \right)^8 \right]^3, \quad (3)$$

where

$$\eta(\tau) := e^{\pi i \tau / 12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) \quad (4)$$

is the Dedekind's eta function and

$$\tau = \frac{1 + \sqrt{-r}}{2}, \quad \tau = \sqrt{-r}, \quad r \text{ positive real.}$$

We have also in the q -notation

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n). \quad (5)$$

In what follows we use the following known result (see Wolfram pages for 'Rogers-Ramanujan Continued Fraction' and [17]):

If

$$R = R(e^{-2\pi\sqrt{r}}),$$

then:

$$j_r = - \frac{(R^{20} - 228R^{15} + 494R^{10} + 228R^5 + 1)^3}{R^5 (R^{10} + 11R^5 - 1)^5} \quad (6)$$

From ([3],[4]) we have

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)} \quad (7)$$

The general hypergeometric function is defined as

$${}_pF_q [\{a_1, a_2, \dots, a_p\}, \{b_1, b_2, \dots, b_q\}, x] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n, \dots, (a_p)_n}{(b_1)_n (b_2)_n, \dots, (b_q)_n} \frac{x^n}{n!}$$

where $(c)_n = c(c+1) \dots (c+n-1)$, hence $(1)_n = n!$.

The standard definition of the elliptic integral of the first kind (see [7],[8],[15]) is:

$$K(x) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - x^2 \sin^2(t)}} \quad (8)$$

$$K(x) = \frac{\pi}{2} {}_2F_1 \left(\left\{ \frac{1}{2}, \frac{1}{2} \right\}; \{1\}; x^2 \right) = \frac{\pi}{2} {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; x^2 \right) \quad (9)$$

In the notation of Mathematica we have

$$K(x) = \text{EllipticK}[x^2] \quad (10)$$

The elliptic singular modulus $k = k_r$ is defined to be the solution of the equation:

$$\frac{K(\sqrt{1-k^2})}{K(k)} = \sqrt{r}. \quad (11)$$

In Mathematica's notation

$$k = k_r = k[r] = \text{InverseEllipticNomeQ}[e^{-\pi\sqrt{r}}]^{1/2}. \quad (12)$$

The complementary modulus is given by $k'_r = \sqrt{1-k_r^2}$. (For evaluations of k_r see [5],[15],[16]).

Also we call $w_r := \sqrt{k_r k_{25r}}$ noting that if one knows $w = w_r$ then (see [2]), knows k_r and k_{25r} .

2 Theorems

Proposition 1. (see [2])

If $q = e^{-\pi\sqrt{r}}$ and r real positive then we define

$$A = A_r := \frac{f^6(-q^2)}{q^2 f^6(-q^{10})} = R(q^2)^{-5} - 11 - R(q^2)^5 \quad (13)$$

then

$$A_r = a_{4r} = \frac{(k_r k'_r)^2}{(w_r w'_r)^2} \left(\frac{w_r}{k_r} + \frac{w'_r}{k'_r} - \frac{w_r w'_r}{k_r k'_r} \right)^3 \quad (14)$$

Theorem 1.

Let a, b, C_1 be constants. One can solve the equation

$$\frac{b^2}{20a} + bX + aX^2 = C_1 X^{5/3}, \quad (15)$$

finding $r > 0$ such that

$$j_r = 250C_1^3 a^{-2} b^{-1}. \quad (16)$$

Then (15) have solution

$$X = \frac{b}{250a} A_r = \frac{b}{250a} \frac{f(-e^{-2\pi\sqrt{r}})^6}{e^{-2\pi\sqrt{r}} f(-e^{-10\pi\sqrt{r}})^6}. \quad (17)$$

Proof.

For to solve the equation (15) find r such that

$$j_r^{1/3} = \frac{5 \cdot 2^{1/3} C_1}{a^{2/3} b^{1/3}} \quad (18)$$

Consider also the transformation of the constants

$$3125m = \frac{b^2}{20a}, \quad 250ml^{-1} = \frac{b^2}{250a} \left(b + \frac{b^2}{20a} \right)^{-1}$$

and

$$ml^{-2} = \frac{b^2}{62500} \left(b + \frac{b^2}{20a} \right)^{-2},$$

with inverse

$$l = \frac{b(20a + b)}{20a}, \quad m = \frac{b^2}{62500a}.$$

Then

$$X = \frac{250m}{l(l - 3125m)} x_1 = \frac{250m}{l - 3125m} x = \frac{b}{250a} x,$$

where x_1 satisfies

$$3125m + 250x_1ml^{-1} + x_1^2ml^{-2} = ml^{-5/3}j^{1/3}x_1^{5/3}$$

If we set $x_1 = lx$, then it is

$$3125 + 250x + x^2 = j^{1/3}x^{5/3}$$

or equivalently

$$3125 + 250A_r + A_r^2 = j_r^{1/3}A_r^{5/3} \quad (19)$$

Relation (19) is equivalent to equation (6), in view of (7). Hence from Proposition 1

$$\begin{aligned} X = A_r = a_{4r} &= \frac{b}{250a} \frac{f(-e^{-2\pi\sqrt{r}})^6}{e^{-2\pi\sqrt{r}}f(-e^{-10\pi\sqrt{r}})^6} = \\ &= \frac{b}{250a} \frac{(k_r k'_r)^2}{(w_r w'_r)^2} \left(\frac{w_r}{k_r} + \frac{w'_r}{k'_r} - \frac{w_r w'_r}{k_r k'_r} \right)^3 = \frac{b}{250a} (R^{-5}(q^2) - 11 - R^5(q^2)) \end{aligned} \quad (20)$$

and the proof is complete.

The j -invariant is connected with the singular modulus from the equation

$$j_r = \frac{256(k_r^2 + k'_r{}^4)^3}{(k_r k'_r)^4}. \quad (21)$$

We can solve (21) and express k_r in radicals to an algebraic function of j_r .

The 5th degree modular equation which connects k_{25r} and k_r is (see [3]):

$$k_r k_{25r} + k'_r k'_{25r} + 2 \cdot 4^{1/3} (k_r k_{25r} k'_r k'_{25r})^{1/3} = 1 \quad (22)$$

We will evaluate the root of (1) first with parametrization and second with Rogers-Ramanujan continued fraction and the Elliptic- K function.

For this it have been showed (see [19]) that if

$$k_{25r} k_r = w_r^2 = w^2, \quad (23)$$

setting the following parametrization of w :

$$w = \sqrt{\frac{L(18 + L)}{6(64 + 3L)}}, \quad (24)$$

we get

$$\frac{(k_{25r})^{1/2}}{w^{1/2}} = \frac{w^{1/2}}{(k_r)^{1/2}} = \frac{1}{2} \sqrt{4 + \frac{2}{3} \left(\frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)^2} + \frac{1}{2} \sqrt{\frac{2}{3} \left(\frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)} \quad (25)$$

where

$$M = \frac{18 + L}{64 + 3L}$$

From the above relations we get also

$$-\frac{k_r - w}{\sqrt{k_r w}} = \frac{k_{25r} - w}{\sqrt{k_{25r} w}} = \sqrt{\frac{2}{3} \left(\frac{L^{1/6}}{M^{1/6}} - 4 \frac{M^{1/6}}{L^{1/6}} \right)} \quad (26)$$

Hence we can consider the above equations as follows: Taking an arbitrary number L we construct an w . Now for this w we evaluate the two numbers k_{25r} and k_r . Thus when we know the w , the k_r and k_{25r} are given from (24),(25),(26). The result is: We can set a number L and from this calculate the two inverse elliptic nome's. But we don't know the r . One can see (from the definition of k_r) that the r can be evaluated from equation

$$r = \frac{K^2(\sqrt{1 - k_r^2})}{K^2(k_r)} \quad (27)$$

Hence we define

$$r = k^{(-1)}(x) := \frac{K^2(\sqrt{1 - x^2})}{K^2(x)} \quad (28)$$

However is very difficult to evaluate the r in a closed form, such as roots of polynomials or else when a number x is given. Some numerical evaluations indicate us that even if x are algebraic numbers, (not trivial as with $k^{(-1)}(2^{-1/2}) = 1$ or the cases $x = k_r$, $r = 1, 2, 3, \dots$) the r are not rational and may even not be algebraics.

The algebraic representation of X

We know that (see [2]):

$$X = X(L) = \frac{b}{250a} \frac{x_L^2(1 - x_L^2)}{(w_L w'_L)^2} \left(\frac{w_L}{x_L} + \frac{w'_L}{\sqrt{1 - x_L^2}} - \frac{w_L w'_L}{x_L \sqrt{1 - x_L^2}} \right)^3 \quad (29)$$

where $x_L = k_r$ is the singular modulus which corresponds to some L .

$$C_1 = \frac{a^{2/3} b^{1/3}}{5 \cdot 2^{1/3}} j_{r_L}^{1/3}. \quad (30)$$

The procedure is to select a number L and from (24),(25) evaluate w_L, x_L and

$$w'_L = \sqrt{\sqrt{1 - \frac{w_L^4}{x_L^2}} \sqrt{1 - x_L^2}}. \quad (31)$$

The solution $X = X(L)$ of (1) is (29) and for this L holds

$$r_L = k^{(-1)}(x_L) \quad (32)$$

$$X = \frac{b}{250a} \frac{f\left(-e^{-2\pi\sqrt{k^{(-1)}(x_L)}}\right)^6}{e^{-2\pi\sqrt{k^{(-1)}(x_L)}} f\left(-e^{-10\pi\sqrt{k^{(-1)}(x_L)}}\right)^6} \quad (33)$$

$$j_{r_L} = 250C_1^3 a^{-2} b^{-1} \quad (34)$$

$$j_{r_L} = \frac{256(x_L^2 + (1 - x_L^2)^2)^3}{x_L^4(1 - x_L^2)^2} = 250 \frac{C_1^3}{a^2 b} \quad (35)$$

Hence we get the next:

Theorem 2.

One can find parametric solutions of (1) if for a given L construct the w_L, x_L and the complementary w'_L (these values are given from (23),(24),(25),(31)). Also $x'_L = \sqrt{1 - x_L^2}$. The C_1 must be

$$C_1 = \sqrt[3]{\frac{a^2 b j_{r_L}}{250}} \quad (36)$$

The solution $X = X_L$ is given from

$$X = X(L) = \frac{b}{250a} \frac{x_L^2(1 - x_L^2)}{(w_L w'_L)^2} \left(\frac{w_L}{x_L} + \frac{w'_L}{\sqrt{1 - x_L^2}} - \frac{w_L w'_L}{x_L \sqrt{1 - x_L^2}} \right)^3 \quad (37)$$

Note.

i) The above solution (37) works for parametric solutions (setting a L), as also for solutions which we know r, k_r and k_{25r} . (For a related method on solving the quintic see Wolfram pages 'Quintic Equation')

ii) In [16] it have been shown that when one knows for some r_0 the k_{r_0} and $k_{r_0/25}$ then can evaluate any $k_{25^n r_0}$ in radicals closed form for all n positive integers. But in general the values k_r and k_{25r} can given from tables or with a simple PC (see [4],[5],[13],[15],[17]).

The inverse functions method

From the analysis in [2], the solution X of (1) can reduced with inverse functions as follows:

Consider the function

$$U(x) = \frac{256(x^2 + (1 - x^2)^2)^3}{x^4(1 - x^2)^2}, \quad (38)$$

The equation $U(x) = t$ have known solution with respect to x , which we will call $x = U^{(-1)}(t)$. Hence

$$\frac{256(k_r^2 + (1 - k_r^2)^2)^3}{k_r^4(1 - k_r^2)^2} = 250 \frac{C_1^3}{a^2 b} \quad (39)$$

or

$$U(k_r) = 250 \frac{C_1^3}{a^2 b}$$

$$k_r = U^{(-1)} \left(250 \frac{C_1^3}{a^2 b} \right)$$

or

$$r = k^{(-1)} \left(U^{(-1)} \left(250 \frac{C_1^3}{a^2 b} \right) \right)$$

The function $k^{(-1)}(x)$ is that of (28).

Theorem 3.

The equation (1) have solution

$$X = \frac{b}{250a} \left(R \left(e^{-2\pi\sqrt{k^{(-1)}(\alpha)}} \right)^{-5} - 11 - R \left(e^{-2\pi\sqrt{k^{(-1)}(\alpha)}} \right)^5 \right) \quad (40)$$

where

$$\alpha = U^{(-1)} \left(250 \frac{C_1^3}{a^2 b} \right) \quad (41)$$

Notes.

- 1) Observe here that we don't need the value of w and the class invariant j .
- 2) From [10] we have

$$R(e^{-x}) = e^{-x/5} \frac{\vartheta_4(3ix/4, e^{-5x/2})}{\vartheta_4(ix/4, e^{-5x/2})}, \forall x > 0$$

Hence the solution can expressed also in theta functions. That is if $\alpha = k_r$, $r = 1, 2, 3, \dots$ then the solution of (1) reduced to that of evaluation of Rogers-Ramanujan continued fraction $R(q)$ with $q = e^{-\pi\sqrt{r}}$. In view of [10] we have

$$X = \frac{b}{250a} \left[e^{2\pi\sqrt{r}} \left(\frac{\vartheta_4(3i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}})}{\vartheta_4(i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}})} \right)^{-5} - 11 - e^{-2\pi\sqrt{r}} \left(\frac{\vartheta_4(3i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}})}{\vartheta_4(i\pi\sqrt{r}/2, e^{-5\pi\sqrt{r}})} \right)^5 \right]$$

Example 1.

The equation

$$X^2 + 3X + \frac{9}{20} = \frac{26}{5\sqrt[3]{3}}X^{5/3}$$

have

$$\alpha = U^{(-1)}\left(\frac{35152}{9}\right) = \frac{\sqrt{3}}{2}$$

hence a solution is

$$X = \frac{3}{250} \left(R \left(e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left(e^{-2\pi\sqrt{r}} \right)^5 \right)$$

where

$$r = \frac{K\left(\frac{1}{2}\right)^2}{K\left(\frac{\sqrt{3}}{2}\right)^2}$$

For this r the X is a solution.

Continuing one can set to

$$X^5 = \frac{b^2}{20aC_1} + \frac{b}{C_1}X^3 + \frac{a}{C_1}X^6 \quad (42)$$

any value $X = X_0$ and $C_1 = 1$ then evaluate

$$a = \frac{-5bX_0^3 + 5X_0^5 + \sqrt{5}\sqrt{4b^2X_0^6 - 10bX_0^8 + 5X_0^{10}}}{10X_0^6} \quad (43)$$

equation (42) holds always and we get that

$$\begin{aligned} R \left(e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left(e^{-2\pi\sqrt{r}} \right)^5 &= \\ &= \frac{25 \left(-5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4} \right)}{bX_0^2} \end{aligned}$$

where $j_r = 250a^{-2}b^{-1}$.

$$j_r = \frac{25000X_0^6}{b \left(-5b + 5X_0^2 + \sqrt{5}\sqrt{4b^2 - 10bX_0^2 + 5X_0^4} \right)^2} \quad (44)$$

The result is the following parametrized evaluation of the Rogers-Ramanujan continued fraction

Theorem 4.

$$A_r = R \left(e^{-2\pi\sqrt{r}} \right)^{-5} - 11 - R \left(e^{-2\pi\sqrt{r}} \right)^5 =$$

$$= \frac{25(-5b + 5t^5 + \sqrt{5}\sqrt{4b^2 - 10bt^2 + 5t^4})}{b} \quad (45)$$

and

$$j_r = \frac{25000t^6}{b(-5b + 5t^2 + \sqrt{5}\sqrt{4b^2 - 10bt^2 + 5t^4})^2} \quad (46)$$

Corollary.

If

$$\sqrt[3]{A_r^2 j_r} = \text{rational}$$

then A_r is of the form

$$A_r = \frac{A + B\sqrt{D}}{C}$$

where A, B, C, D rationals

Theorem 5.

If for a certain $r > 0$ we know the value of $R(e^{-\pi\sqrt{r}})$ in radicals, then we can evaluate both k_r and k_{25r} and the opposite.

Proof.

Suppose we know for a certain $r > 0$ the value of $R(e^{-\pi\sqrt{r}})$, (the correspondence between $R(e^{-\pi\sqrt{r}})$ and $R(e^{-2\pi\sqrt{r}})$ is given by (96) bellow or see [13]). Then from (6) we know the value of j_r and from (21) we know k_r . Let also $q = e^{-\pi\sqrt{r}}$, $r > 0$ and $v_r = R(q)$, then it have been proved by Ramanujan that

$$v_{r/25}^5 = v_r \frac{1 - 2v_r + 4v_r^2 - 3v_r^3 + v_r^4}{1 + 3v_r + 4v_r^2 + 2v_r^3 + v_r^4},$$

Hence we can get the value of $R(e^{-\pi\sqrt{r/5}})$. Hence again from (6) we find $j_{r/25}$ and from (21) the value of $k_{r/25}$. But from relation (53) bellow knowing k_r and $k_{r/25}$ we can evaluate all $k_{25^n r}$, $n = 1, 2, \dots$ and consequently k_{25r} as a special case. The opposite follow from Proposition 1.

Theorem 6.

The solution U_0 of the equation

$$U_0 = j_r^{1/3} \left(125 - \sqrt{12500 + U_0} \right)^{5/3} \quad (47)$$

is

$$U_0 = U(j_r) = \sum_{n=1}^{\infty} \frac{j_r^{n/3}}{n!} \left[\frac{d^{n-1}}{da^{n-1}} \left(125 - \sqrt{12500 + a} \right)^{5n/3} \right]_{a=0}$$

If $X = x_0$ is root of

$$U_0 = X^2 + 250X + 3125$$

then

$$3125 + 250x_0 + x_0^2 = j_r^{1/3}(-1)^{1/3}x_0^{5/3} \quad (47a)$$

and

$$x_0 = X = X_r = A_r = R(e^{-2\pi\sqrt{r}})^{-5} - 11 - R(e^{-2\pi\sqrt{r}})^5 \quad (47b)$$

Proof.

Consider (1), then make the change of variable $U_0 = X^2 + 250X + 3125$, we arrive to (47). The Legendre inversion theorem states that the solution of $y = af(y)$ (see [7] pg.132-133) is

$$y = \sum_{n=1}^{\infty} \frac{a^n}{n!} \left[\frac{d^{n-1}}{dx^{n-1}} f(x)^n \right]_{x=0}$$

This theorem works for j_r small, for example with $j_1 = 1728$ it converges very slowly but for r such that $j_r = 800$, (r -complex) we get numerical evaluations and hence also theoretical.

Theorem 7.

If

$$c_n := \left[\frac{d^{n-1}}{da^{n-1}} (125 - \sqrt{12500 + a})^{5n/3} \right]_{a=0}$$

then

$$c_n = \frac{5^6}{3} (-1)^{n+1} n \cdot 10^{-5n/3} {}_2F_1 \left[\frac{5n}{6}, \frac{5n+3}{6}; \frac{2(n+3)}{3}; \frac{1}{5} \right] \frac{\Gamma(5n/3)}{\Gamma(2+2n/3)}$$

Proof.

Recall a theorem of Euler (see [18] pg.306-307). If the root of

$$aqx^p + x^q = 1$$

is x , then

$$x^n = \frac{n}{q} \sum_{k=0}^{\infty} \frac{\Gamma(\{n+pk\}/q) (-qa)^k}{\Gamma(\{n+pk\}/q - k + 1) k!}$$

Hence from the fact that

$$w = \frac{250(125 - \sqrt{12500 + x})}{3125 - x}$$

is solution of

$$acb^{-2}w^2 + w = 1, \quad a = 1, \quad b = -250, \quad c = -3125 + x$$

we get

$$(-125 + \sqrt{12500 + x})^n = \frac{n}{250^n} \sum_{k=0}^{\infty} \frac{\Gamma(n+2k) (-1)^k}{\Gamma(n+k+1) 62500^k k!} (-3125+x)^{k+n} \quad (48)$$

Using the formula

$$f^{(\nu)}(x_0) = \sum_{n=0}^{\infty} \frac{f^{(\nu+n)}(x_0)}{n!} (-x_0)^n$$

the result follows.

Theorem 7 is for numerical evaluations since the hypergeometric series can more easily be computed than the $(n-1)$ th derivative of the $5n/3$ power of $125 - \sqrt{12500 + x}$.

Corollary.

For every 'suitable' value of x_0 such that $X_r = x_0 : (a)$, X_r is of (47b) exists a r solution of (a) such that

$$\begin{aligned} & X_r^2 + 250X_r + 3125 = \\ & = 3^{-1} \cdot 5^6 \sum_{n=1}^{\infty} (-1)^{n+1} n \frac{\Gamma(5n/3)}{\Gamma(2+2n/3)} {}_2F_1 \left[\frac{5n}{6}, \frac{5n+3}{6}; \frac{2(n+3)}{3}; \frac{1}{5} \right] \frac{(10^{-5}j_r)^{n/3}}{n!} \end{aligned}$$

Example 2.

For $X_r = x_0 = -12$, we have $r = -0.186710441... - i0.251574161...$ and

$$X_r^2 + 250X_r + 3125 = 269 = \sum_{n=1}^{\infty} \frac{(-j_r)^{n/3}}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} (125 - \sqrt{12500 + z})^{5n/3} \right]_{z=0}$$

Example 3.

Consider the equation

$$X^2 + 250X + 3125 = 2(-1)^{1/3} 10^{2/3} X^{5/3}$$

Then clearly $j = j_r = 800$ and a solution is

$$X = x_0 = -125 + \sqrt{12500 + \sum_{n=1}^{\infty} \frac{(2\sqrt[3]{100})^n}{n!} \left[\frac{d^{n-1}}{dz^{n-1}} (125 - \sqrt{12500 + z})^{5n/3} \right]_{z=0}}$$

3 Applications

Example 4.

Set $L = 1/3$ then

$$w_N = w_1(L) = w_1 \left(\frac{1}{3} \right) = \frac{1}{3} \sqrt{\frac{11}{78}}$$

and

$$\begin{aligned} & k_N = x_L = x_1(L) = x_1 \left(\frac{1}{3} \right) = \\ & = \frac{\frac{1}{3} \sqrt{\frac{11}{78}}}{\left(\frac{-4(\frac{11}{13})^{1/6} + (\frac{13}{11})^{1/6}}{\sqrt{6}} + \frac{1}{2} \sqrt{4 + \frac{2}{3} \left(-4 \left(\frac{11}{13} \right)^{1/6} + \left(\frac{13}{11} \right)^{1/6} \right)^2} \right)^2} \end{aligned}$$

and

$$k_{25N} = x_2(L) = x_2\left(\frac{1}{3}\right) =$$

$$= \frac{1}{3}\sqrt{\frac{11}{78}}\left(\frac{-4\left(\frac{11}{13}\right)^{1/6} + \left(\frac{13}{11}\right)^{1/6}}{\sqrt{6}} + \frac{1}{2}\sqrt{4 + \frac{2}{3}\left(-4\left(\frac{11}{13}\right)^{1/6} + \left(\frac{13}{11}\right)^{1/6}\right)^2}\right)^2$$

where the N is given by

$$N = r_L = r_{1/3} = \frac{K^2\left(\sqrt{1 - x_1\left(\frac{1}{3}\right)^2}\right)}{K^2\left(x_1\left(\frac{1}{3}\right)\right)}$$

From the value of x_L we obtain j_{r_L} and hence the corresponding C_1 in radicals-closed form and hence $X = X_L$ from (37) and (31). The numbers a, b take arbitrary values.

We note here that in future application of this method one must tabulate values of (r, w_r) and not j_r or k_r which follow from these of w_r . This can be done in some cases using the Main Theorem in [16] and the solution (37) of Theorem 2 of the present paper.

Form [16] we have if

$$Q(x) = \frac{\left(-1 - e^{\frac{1}{5}y} + e^{\frac{2}{5}y}\right)^5}{\left(e^{\frac{1}{5}y} - e^{\frac{2}{5}y} + 2e^{\frac{3}{5}y} - 3e^{\frac{4}{5}y} + 5e^y + 3e^{\frac{6}{5}y} + 2e^{\frac{7}{5}y} + e^{\frac{8}{5}y} + e^{\frac{9}{5}y}\right)} \quad (49)$$

$$y = \operatorname{arcsinh}\left(\frac{11+x}{2}\right)$$

$$Y = U_0(X) = \sqrt{-\frac{5}{3X^2} + \frac{25}{3X^2h(X)} + \frac{X^4}{h(X)} + \frac{h(X)}{3X^2}} \quad (50)$$

$$h(x) = \left(-125 - 9x^6 + 3\sqrt{3}\sqrt{-125x^6 - 22x^{12} - x^{18}}\right)^{1/3}$$

$$U_1(Y) = X = \sqrt{-\frac{1}{2Y^2} + \frac{Y^4}{2} + \frac{\sqrt{1 + 18Y^6 + Y^{12}}}{2Y^2}}. \quad (51)$$

and

$$P(x) = P[x] = U_0[Q^{1/6}[U_1[x]]] \text{ and } P^{(n)}(x) = (P \underbrace{\circ \dots \circ}_n P)(x) \quad (52)$$

then

$$k_{25^{n_{r_0}}} = \sqrt{1/2 - 1/2\sqrt{1 - 4\left(k_{r_0}k'_{r_0}\right)^2\prod_{j=1}^n P^{(j)}\left[\sqrt[12]{\frac{k_{r_0}k'_{r_0}}{k_{r_0/25}k'_{r_0/25}}}\right]^{24}}} \quad (53)$$

Example 5.

$$\begin{aligned}
k_{1/5} &= \sqrt{\frac{9 + 4\sqrt{5} + 2\sqrt{38 + 17\sqrt{5}}}{18 + 8\sqrt{5}}} \\
k_5 &= \sqrt{\frac{9 + 4\sqrt{5} - 2\sqrt{38 + 17\sqrt{5}}}{18 + 8\sqrt{5}}} \\
k_{125} &= \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - (9 - 4\sqrt{5})P[1]^2}}
\end{aligned} \tag{54}$$

Example 6.

It is

$$\begin{aligned}
k_1 &= \frac{1}{\sqrt{2}} \\
k_{25} &= \frac{1}{\sqrt{2 \left(51841 + 23184\sqrt{5} + 12\sqrt{37325880 + 16692641\sqrt{5}} \right)}}
\end{aligned}$$

Hence

$$k_{625}k'_{625} = \frac{1}{2(161 + 72\sqrt{5})} P[161 - 72\sqrt{5}]$$

and hence

$$k_{625} = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \left(\frac{P[161 - 72\sqrt{5}]}{161 + 72\sqrt{5}} \right)^2}} \tag{55}$$

By this way we can evaluate every k_r which is $r = 4^l 9^m 25^n r_0$ when k_{r_0} and $k_{r_0/25}$ are known, $l, m, n \in \mathbf{N}$.

Note. In the case that

$$\left(\frac{L}{M} \right)^{1/6} = A = \frac{f}{g} = 3^a \frac{2p+1}{2h+1}, \tag{56}$$

where f, g positive integers, with $f < g$ and $p, h \not\equiv 0 \pmod{4}$, ($a \in \mathbf{Z} - \{-1, 0, 1\}$), we can find w from a given $x = k_r$ which is of the form

$$x = k_r = \frac{t_1 \sqrt{t_2}}{(t_3 + \sqrt{t_4})^2}$$

where $t_i, i = 1, 2, 3, 4$ rationals.

In view of (25) and the action of the command recognize, (which is needed to put number x into his form) the output will be an octic equation with step 2 containing nested square roots:

$$<< \text{NumberTheory}' \text{Recognize}'$$

`Solve[Reduce[N[x,1000],16,v]==0,v]`

The smallest root it will be

$$\sqrt{D} = \sqrt{4096 + 88 \left(3^a \frac{2p+1}{2h+1} \right)^6 + \left(3^a \frac{2p+1}{2h+1} \right)^{12}}$$

One can see that for these x 's the f and g are given from the Diofantie equation

$$9D = g^{12} \left(4096 + 88 \frac{f^6}{g^6} + \frac{f^{12}}{g^{12}} \right) \quad (57)$$

hence the number A will be known and

$$w^2 = \frac{4096 - 20A^6 + A^{12} + (-64 + A^6) \sqrt{4096 + 88A^6 + A^{12}}}{108A^6} \quad (58)$$

Hence we have the value of X in radicals.

If for example

$$x = k_r = \frac{-37754085\sqrt{3} + 3\sqrt{476791023769787}}{77\sqrt{2}(-435 + \sqrt{224799})^2}$$

then for all C_1, a, b such that

$$j_r = \frac{256(x^2 + (1 - x^2)^2)^3}{x^4(1 - x^2)^2} = 250 \frac{C_1^3}{a^2b}$$

with Mathematica and the package 'Recognize' we evaluate

$$\text{Solve}[\text{Recognize}[N[x, 1000], 16, v] == 0, v]$$

which gives the value of x in the desired form. The solution that corresponds to x have smallest square root

$$\sqrt{D} = \sqrt{1430373071309361}$$

the command 'Reduce' give us the f and g

$$\text{Reduce}[9D == (4096 + 88(f/g)^6 + (f/g)^{12})g^{12}, \{f, g\}, \text{Integers}]$$

Hence we get the values $f = 7, g = 11$ and w . The solution (29) is

$$X = \frac{A}{35153041^3} [5579801448 - 11724990\sqrt{224799} + \sqrt{6362897839(9487950991 - 20011160\sqrt{224799})}]^3$$

where

$$\begin{aligned}
A &= A_1 + B_1 - \frac{1}{2}\sqrt{A_2 + B_2\sqrt{224799}} \\
A_1 &= \frac{93573266991461291403517623659291588}{29148873138738228269392700625} \\
B_1 &= \frac{572443990137\sqrt{15689769027087558724590525868357405909071}}{22336301255738105953557625} \\
A_2 &= \frac{70047382179949155201445598248892391809778079434882871423635278599557376}{849656805258255011387371218620407164266412150030875390625} \\
B_2 &= \frac{12578872085638673940246389099496155002610886190948544424950380544}{72342001299127714890367919848480814326642158367890625}
\end{aligned}$$

Theorem 8.

When $(L/M)^{1/6}$ is rational, we can always find k_{25r} from k_r .

Example 7.

Set $a = (k_r k'_r)^2$, $b = (ww')^2$, then equation (1) have solution

$$X = \frac{m_5^3}{250},$$

where m_5 is the multiplier (see [3]).

Hence

$$3125 \frac{(ww')^4}{(k_r k'_r)^2} + 250(ww')^2 m_5^3 + (k_r k'_r)^2 m_5^6 = (k_r k'_r)^{4/3} (ww')^{2/3} j_r^{1/3} m_5^5$$

Example 8.

We will find a solution of the equation

$$\frac{3125}{16} + 125X + 4X^2 = 132X^{5/3} \quad (59)$$

in radicals using Theorem 3.

Solution.

It is $a = 4$, $b = 125$, and we have to solve $j_r = 287496$ or equivalently $r = 4$.

Hence a solution of (59) is:

$$X = \frac{125e^{4\pi}}{250 \cdot 4} \frac{f(-e^{-4\pi})^6}{f(-e^{-20\pi})^6} = \frac{1}{8} (R(e^{-4\pi})^{-5} - 11 - R(e^{-4\pi})^5) \quad (60)$$

The exact root in radicals can be found but is very large and complicated with our method. We give a way how one can obtain it:

It is known that

$$R(e^{-2\pi}) = -\frac{1+\sqrt{5}}{2} + \sqrt{\frac{5+\sqrt{5}}{2}} \quad (61)$$

But from the duplication formula (see [4],[13]):
If $u = R(q)$ and $\nu = R(q^2)$, then

$$\frac{\nu - u^2}{\nu + u^2} = u\nu^2. \quad (62)$$

Hence we find the value of $R(e^{-4\pi})$ in radicals and hence the solution of (59) using (60),(61),(62).

The root using the program Mathematica is

$$X = \frac{143375}{16} + \frac{64125\sqrt{5}}{16} + \frac{1}{2}\sqrt{\frac{20553203125}{32} + \frac{9191671875\sqrt{5}}{32}}$$

In this case it is more convenient to use Mathematica's command Solve. But in other cases these solutions can not found.

From the above result we have shown that

$$\begin{aligned} & \frac{1}{8} (R(e^{-4\pi})^{-5} - 11 - R(e^{-4\pi})^5) = \\ & = \frac{143375}{16} + \frac{64125\sqrt{5}}{16} + \frac{1}{2}\sqrt{\frac{20553203125}{32} + \frac{9191671875\sqrt{5}}{32}}. \end{aligned}$$

One can see that if we set

$$Y_\tau := \frac{b}{250a} \left(R(e^{2\pi i\tau})^{-5} - 11 - R(e^{2\pi i\tau})^5 \right) \quad : (a)$$

Then if

$$\tau = \frac{1 + \sqrt{-r}}{2} \text{ or } \tau = \sqrt{-r},$$

where r positive integer in some cases we can evaluate Y_τ solving directly the equation (1), with parameters $a = 4$, $b = 125$ and C_1 depended on j_τ . Some examples are

$$\begin{aligned} & \frac{1}{16} \left[R(e^{\pi(i-\sqrt{51})})^{-5} - 11 - R(e^{\pi(i-\sqrt{51})})^5 \right] = \\ & = -\frac{125}{16} [5541103 + 1343914\sqrt{17} + \sqrt{61407604829690 + 14893531819350\sqrt{17}} + \\ & + 4\sqrt{\{ \frac{1}{6140760482969 + 1489353181935\sqrt{17}} (94272348104055803848937570 + \\ & + 22864402871059934148609270\sqrt{17} + \\ & + 2063164169063100077\sqrt{170(6140760482969 + 1489353181935\sqrt{17})} + \\ & + 8506643792036854023\sqrt{61407604829690 + 14893531819350\sqrt{17}} \}}]. \end{aligned} \quad (63)$$

$$Y_{\sqrt{-1/5}} = \frac{5\sqrt{5}}{8}. \quad (64)$$

$$Y_{\sqrt{-2/5}} = \frac{5}{8} (5 + 2\sqrt{5}). \quad (65)$$

$$Y_{\sqrt{-3/5}} = \frac{5}{16} (25 + 11\sqrt{5}). \quad (66)$$

$$Y_{\sqrt{-4/5}} = \frac{5}{16} (25 + 13\sqrt{5} + 5\sqrt{58 + 26\sqrt{5}}). \quad (67)$$

$$Y_{\sqrt{-5/5}} = \frac{125}{8} (2 + \sqrt{5}). \quad (68)$$

$$Y_{\sqrt{-6/5}} = \frac{5}{8} (50 + 35\sqrt{2} + 3\sqrt{5(99 + 70\sqrt{2})}). \quad (69)$$

$$Y_{\sqrt{-9/5}} = \frac{5}{8} (225 + 104\sqrt{5} + 10\sqrt{1047 + 468\sqrt{5}}). \quad (70)$$

$$Y_{\sqrt{-12/5}} = \frac{5}{16} (1690 + 975\sqrt{3} + 29\sqrt{6755 + 3900\sqrt{3}}). \quad (71)$$

$$Y_{\sqrt{-14/5}} = \frac{5}{8} (1850 + 585\sqrt{10} + 7\sqrt{5(27379 + 8658\sqrt{10})}). \quad (72)$$

$$Y_{\sqrt{-17/5}} = \frac{5}{8} (5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}}). \quad (73)$$

We describe the method bellow.

For some r positive rational we find the value of $j_{r/5}$; this can be done with the command 'Recognize' of the program Mathematica (if $j_{r/5}$ is root of a small degree algebraic polynomial equation). Then we find C_1 (from (16)) and for the values $a = 4$, $b = 125$ there will be

$$Y_r = \text{root of equation (1)}.$$

In many cases of such r , equation (1) can solved in radicals with Mathematica (we have not find the reason yet), but still in others not. Hence we get relations like (63)–(73).

4 More Theorems and Results

Theorem 9. (Conjecture)

For every positive real r , we have

$$Y_{\sqrt{-r/5}} Y_{\sqrt{-r^{-1}/5}} = \frac{125}{64}. \quad (74)$$

If l, m, t and d are integers and

$$Y_{\sqrt{-r/5}} = \frac{l + m\sqrt{d}}{t} \quad (75)$$

then

$$l^2 - m^2d = t^2 \frac{125}{64} \quad (76)$$

In general we conjecture that

Theorem 10. (Conjecture)

If $r = a_1/b_1$ with $a_1, b_1 \in \mathbf{N}$ and $\text{GCD}(a_1, 5) = 1$, $\text{GCD}(b_1, 5) = 1$ then

$$\deg(Y_{\sqrt{-r/5}}) = \deg(j_{\sqrt{-r/5}}) \quad (77)$$

For example if $\deg(Y_{\sqrt{-r/5}}) = 4$, then

$$Y_{\sqrt{-r/5}} = A + B\sqrt{D} \quad (78)$$

where $\deg(A) = \deg(D) = 2$ and

$$A^2 - B^2D = \frac{125}{64}U \quad (79)$$

where $\deg(U) = 2$ or $U = 1$. If $U \neq 1$ then $U = l + m\sqrt{d}$ and also if $j_{\sqrt{-r/5}}$ have smallest nested square root \sqrt{d} , then $UU^* = l^2 - m^2d = 1$. The symbol $*$ denotes the algebraic conjugate.

Hence for example if $r = 6$ then $d = 2$ and

$$j_{\sqrt{-6/5}} = 8640[25551735275 - 18067805280\sqrt{2} - \\ -196\sqrt{10(3399058140008707 - 2403497060447490\sqrt{2})}]$$

then $U = l + m\sqrt{2}$ with

$$l^2 - 2m^2 = 1.$$

We solve the above Pell's equation. The solution we looking for, taking the smallest to higher order solutions, for this example with $r = 6$ is $l_1 = 99$ and $m_1 = 70$. Hence $A^2 - B^2D = \frac{125}{64}(99 + 70\sqrt{2})$.

Now we assume that $A = k_1 + l_1\sqrt{d}$, again with $d = 2$ and $D = k_2 + l_2\sqrt{d}$, etc... We proceed solving Pell's equations.

Theorem 11.

For a given $r \in \mathbf{N}$ and $\deg(Y_{\sqrt{-r/5}}) = 2, 4$, or 8 , if the smallest nested root of

$j_{\sqrt{-r/5}}$ is \sqrt{d} then we can evaluate the Rogers-Ramanujan continued fraction with integer parameters.

i) In the case $\deg(Y_{\sqrt{-r/5}}) = 2$ then

$$Y_{\sqrt{-r/5}} = \frac{l + m\sqrt{d}}{t} \quad (80)$$

where

$$l^2 - m^2d = 1 \text{ and } l, m, d \in \mathbf{N} \quad (81)$$

ii) In the case $\deg(Y_{\sqrt{-r/5}}) = 4$ we have

a) If $U \neq \frac{125}{64}$, then

$$Y_{\sqrt{-r/5}} = \frac{5}{8} \sqrt{\left(a_0 + \sqrt{-1 + a_0^2}\right) \left(\sqrt{5+p} - \sqrt{p}\right)} \quad (82)$$

where

$$Y_{\sqrt{-r/5}} Y_{\sqrt{-r/5}}^* = \frac{125}{64} \left(a_0 + \sqrt{a_0^2 - 1}\right), \quad (83)$$

with, a_0 positive integer, is solution of $l^2 - m^2d = 1$. Hence $l = a_0$ and $m = d^{-1/2} \sqrt{a_0^2 - 1}$ is positive integer. The parameter p is positive rational can be found from the numerical value of $Y_{\sqrt{-r/5}}$.

b) If $U = \frac{125}{64}$, then

$$Y_{\sqrt{-r/5}} = A + \frac{1}{8} \sqrt{-125 + 64A^2}, \quad (84)$$

where we set $A = k + l\sqrt{d}$. Then a starting point for the evaluation of the integers k, l will be the relation

$$l^2 = \frac{(A - k)^2}{d} = \text{square of integer} \quad (85)$$

iii) If $\deg(Y_{\sqrt{-r^4-15-1}}) = 4$, then we can evaluate $Y_{\sqrt{-r^5-1}}$.

It holds $\deg(Y_{\sqrt{-r^5-1}}) = 8$, the minimal polynomial of $Y_{\sqrt{-r^5-1}}/Y_{\sqrt{-r^4-15-1}}$ is of degree 4 or 8 and symmetric. Hence it can be reduced in at most 4th degree polynomial, hence it is solvable. Thus it remains the evaluation of $Y_{\sqrt{-r^4-15-1}}$, which can be done with the help of step (ii).

$$Y_{\sqrt{-r^5-1}} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2}} \left(\sqrt{p+5} - \sqrt{p}\right) 2^{-1} (\sqrt{x+4} - \sqrt{x}) \quad (86)$$

where $x = a_1 + b_1\sqrt{d} + c\sqrt{a_2 + b_2\sqrt{d}}$, a_1, b_1, a_2, b_2, c integers and

$$Y_{\sqrt{-r^5-14-1}} = \frac{5}{8} \sqrt{a_0 + \sqrt{-1 + a_0^2}} \left(\sqrt{p+5} - \sqrt{p}\right)$$

Example 9.

For $r = 68 = 4 \cdot 17$ and from (73) we have $d = 85$

$$\begin{aligned} x &= a_1 + b_1\sqrt{85} + c\sqrt{a_2 + b_2\sqrt{85}} \\ Y_{\sqrt{-68/5}}/Y_{\sqrt{-17/5}} &= 2^{-1} (\sqrt{x+4} - \sqrt{x}) \\ a_1 &= 2891581250, b_1 = 313636050, c = 12960 \\ a_2 &= 99557521554, b_2 = 10798529365 \end{aligned}$$

hence

$$\begin{aligned} Y_{\sqrt{-68/5}} &= Y_{\sqrt{-17/5}} 2^{-1} (\sqrt{x+4} - \sqrt{x}) = \\ &= \frac{5}{16} \left(5360 + 585\sqrt{85} + 4\sqrt{3613670 + 391950\sqrt{85}} \right) (\sqrt{x+4} - \sqrt{x}) \end{aligned}$$

Theorem 12.

If $r = a_1/b_1$ with $\deg(j_{r/5}) = \nu \leq 4$, then $\deg(A_{r/5}) = \nu$ and equation (1) (with a, b rationals) can be solved in radicals.

Application.

If $r = 3/4$ then $\deg(j_{3/20}) = 4$ and $A_{r/5}$ is solution of

$$15625 - 2112500v + 443375v^2 - 16900v^3 + v^4 = 0$$

hence

$$\begin{aligned} A_{3/20} &= R \left(e^{-\pi\sqrt{3/5}} \right)^{-5} - 11 - R \left(e^{-\pi\sqrt{3/5}} \right)^5 = \\ &= \frac{5}{2} \left(1690 - 975\sqrt{3} + 29\sqrt{6755 - 3900\sqrt{3}} \right) \end{aligned}$$

Theorem 13.

If $Q(x) := x^5$ then

$$\frac{1}{j_\tau^{1/3}} \left[R \left(e^{2\pi i \tau} \right)^{-5} - 11 - R \left(e^{2\pi i \tau} \right)^5 \right]^{1/3} = \sqrt[3]{\frac{-125}{j_\tau} + \sqrt{\frac{12500}{j_\tau^2} + Q \left(\sqrt[3]{\frac{-125}{j_\tau}} + \dots \right)}} \quad (87)$$

Proof.

Equation (1) for $a = 1, b = 250j_\tau^{-1}, C_1 = 1$ can be written in the form

$$(X^3 - a_1)^2 - b_1 = X^5 + c_1, \quad (88)$$

where $a_1 = -125j_\tau^{-1}, b_1 = 12500j_\tau^{-2}, c_1 = 0$

Hence Y_τ we can be expressed in nested periodical functions. This completes the proof.

Example 10.

If

$$C_1^3 = 32a^2b$$

then

$$X = \frac{b}{250a} \left(R(e^{-2\pi\sqrt{2}})^{-5} - 11 - R(e^{-2\pi\sqrt{2}})^5 \right)$$

Equation (1) and the Derivative of Rogers-Ramanujan Continued fraction

From [11] it is known that if

$$N(q) = q^{5/6} f(-q)^{-4} \frac{R'(q)}{R(q)} \quad (89)$$

and $N(q^2) = u(q) = u$, $N(q^3) = h(q) = h$ and $N(q) = v(q) = v$, then

$$5u^6 - u^2v^2 - 125u^4v^4 + 5v^6 \stackrel{?}{=} 0 \quad (90)$$

and

$$125h^{12} + h^3v^3 + 1125h^9v^3 + 1125h^3v^9 + 1953125h^9v^9 - 125v^{12} \stackrel{?}{=} 0 \quad (91)$$

which are solvable. But from [12] we have

$$\frac{5R'(q)}{R(q)(R(q)^{-5} - 11 - R(q)^5)^{1/6}} = f^4(-q)q^{-5/6} \quad (92)$$

or

$$N(q) = \frac{1}{5} (R(q)^{-5} - 11 - R(q)^5)^{1/6} \quad (93)$$

Hence the solution of (1) can also given in the form

$$X = X_r = \frac{125b}{2a} N(q^2)^6 \quad (94)$$

and from (85) we have

$$2^{2/3}a^{1/3}b^{1/3}(X_rX_{4r})^{1/3} + \frac{10 \cdot 2^{1/3}a}{b^{1/3}}(X_rX_{4r})^{2/3} - 2a^{2/3}(X_r + X_{4r}) = 0 \quad (95)$$

Note.

One can prove relation (90) using (89),(93) and the duplication formula (see [13]):

$$\frac{R(q^2) - R^2(q)}{R(q^2) + R^2(q)} = R(q)R^2(q^2) \quad (96)$$

The same method can work and with other higher modular equations of the derivative but the evaluations are very difficult even for a program.

Another interesting note that can simplify the problem is the singular moduli of the fifth base (see [14],[15]):

$$u(x) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; x\right). \quad (97)$$

In this case we have

$$j_r = \frac{432}{\beta_r(1 - \beta_r)} = \frac{250C_1^3}{a^2b}, \quad (98)$$

where β_r is the solution of

$$\frac{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; 1 - \beta_r\right)}{{}_2F_1\left(\frac{1}{6}, \frac{5}{6}; 1; \beta_r\right)} = \sqrt{r} \quad (99)$$

The moduli β_r can be evaluated from k_r and the opposite from the relation

$$\frac{256(k_r^2 + (1 - k_r^2)^2)^3}{k_r^4(1 - k_r^2)^2} = \frac{432}{\beta_r(1 - \beta_r)} \quad (100)$$

Proposition 2.

The equation (1) have solution

$$X = \frac{b}{250a} \left(R \left(e^{-2\pi\sqrt{\beta^{(-1)}(\alpha)}} \right)^{-5} - 11 - R \left(e^{-2\pi\sqrt{\beta^{(-1)}(\alpha)}} \right)^5 \right), \quad (101)$$

where

$$\alpha = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{216a^2b}{125C_1^3}}, \quad \beta^{(-1)}(x) = \left(\frac{u(1-x)}{u(x)} \right)^2.$$

Corollary.

The equation

$$aX^2 + bX + \frac{b^2}{20a} = \frac{6a^{2/3}b^{1/3}}{5\beta_r^{1/3}(1 - \beta_r)^{1/3}} X^{5/3}$$

admits solution $X = A_r$.

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